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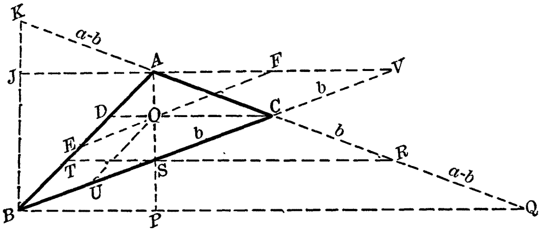
In the usual form of the law of the mean for derivatives, $f(x+h) - f(x) = hf'(x+\theta h)$, the quantity θ depends on both x and h. It is natural to inquire into the character of the functional dependence of θ on x and h. A general investigation of this dependence has been made by Hedrick, *Annals of Mathematics,* series 2, p. 177. Professor Downing in the second discussion undertakes the more special task of studying the nature of θ for simple forms of the function $f(x)$. He takes up the cases of linear, quadratic, and cubic functions.

I. Geometrical Proofs of the Law of Tangents.

By W. V. Lovitt, Colorado College.

The geometrical proof depends in many cases upon finding pairs of lines the ratio of whose lengths is $(a-b)/(a+b)$, where a and b are sides of a given triangle. We proceed then to find such pairs of lines.



Make

$$AC = SC = CV = CR = b;$$

make

$$CK = CQ = CB = a.$$

Then

$$AK = BS = RQ = a - b,$$

and

$$BV = RK = a + b.$$

Draw CD bisecting the angle ACB. The lines VJ, TR, CD, and BQ are parallel. Draw AP and $BK \perp BQ$. Draw $EOF \parallel BC$ and $OU \parallel AB$.

Then

$$BU = \tfrac{1}{2}(a-b), \qquad UC = \tfrac{1}{2}(a+b),$$

and

$$\frac{a-b}{a+b} = \frac{BS}{BV} = \frac{EO}{EF} = \frac{BU}{UC} = \frac{DO}{OC} = \frac{TS}{AV} = \frac{TS}{SR} = \frac{BP}{PQ} = \frac{JA}{JV} = \frac{BS}{AQ} = \frac{AK}{BV}. \tag{1}$$

We also find

$$\tfrac{1}{2}(A+B) = \angle OSC = \angle OAC = \angle JBV = \angle JKA,$$

$$\tfrac{1}{2}(A-B) = \angle BAP = \angle JBA.$$

In many geometrical proofs of the law of tangents the only points of difference are in the precise figure and method used in establishing some one or more of the equalities (1). If equalities (1) were a well known part of the course in elementary geometry, then the several different proofs of the law of tangents would be simpler.

We give below some proofs which are believed to be new. With one exception they make use of the figure above. In each instance only such part of the figure need be actually drawn as is used in the proof.

(1)
$$\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\sin BSA}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)},$$

$$\frac{a+b}{c}=\frac{\sin ABQ}{\sin AQB}=\frac{\cos BAP}{\sin C/2}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)},$$

whence

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

(2)
$$\tan\tfrac{1}{2}(A-B)=\frac{BP}{AP}=\frac{(a-b)\sin BSP}{(a+b)\cos PAQ}=\frac{(a-b)\sin\frac{1}{2}(A+B)}{(a+b)\cos\frac{1}{2}(A+B)}.$$

$$=\frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B).$$

(3)
$$\angle SAC=\tfrac{1}{2}(A+B);\qquad \angle SAB=\tfrac{1}{2}(A-B);$$

$$EF\parallel BV;\qquad CD\perp AS;\qquad CD\parallel AV.$$

$$CS=CA=CV=b;\qquad BS=a-b;\qquad BV=a+b;$$

$$\tan\tfrac{1}{2}(A-B)=\frac{DO}{AO};\qquad \tan\tfrac{1}{2}(A+B)=\frac{CO}{AO};$$

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{DO}{CO}=\frac{DO}{AF}=\frac{EO}{EF}=\frac{BS}{BV}=\frac{a-b}{a+b}.$$

(4)
$$\angle VBJ=\tfrac{1}{2}(A+B);\qquad \angle ABJ=\tfrac{1}{2}(A-B);$$

$$\angle VJB=\angle SAV=90^\circ;\qquad CA=CS=CV=b;$$

$$\tan\tfrac{1}{2}(A-B)=\frac{AJ}{BJ};\qquad \tan\tfrac{1}{2}(A+B)=\frac{VJ}{BJ};$$

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{AJ}{VJ}=\frac{BS}{BV}=\frac{a-b}{a+b},$$

or

$$\tan\tfrac{1}{2}(A-B)=\frac{AJ}{BJ}=\frac{BS\sin\frac{1}{2}(A+B)}{BV\cos\frac{1}{2}(A+B)}$$

$$=\frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B).$$

(5)
$$\angle ABJ=\tfrac{1}{2}(A-B);\qquad \angle JKA=\tfrac{1}{2}(A+B);\qquad \angle AJK=90^\circ;$$

$$CA=CV=b;\qquad AK=a-b;$$

$$\tan \tfrac{1}{2}(A - B) = \frac{AJ}{BJ}; \qquad \tan \tfrac{1}{2}(A + B) = \frac{VJ}{BJ};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{AJ}{VJ} = \frac{AK}{BV} = \frac{a - b}{a + b};$$

or

$$\tan \tfrac{1}{2}(A - B) = \frac{AJ}{BJ} = \frac{AK \sin \frac{1}{2}(A + B)}{BV \cos \frac{1}{2}(A + B)} = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B),$$

(6) Circumscribe a circle about ABC. Draw CP bisecting the angle C. Make $CR = CA = b$. Then $BR = a - b$.

$$\angle\, CPS = \angle\, CAR = \tfrac{1}{2}(A + B),$$

$$\angle\, BPS = \angle\, BAR = \tfrac{1}{2}(A - B),$$

$$\angle\, BQS = \angle\, RQS = \angle\, BCP = \frac{C}{2},$$

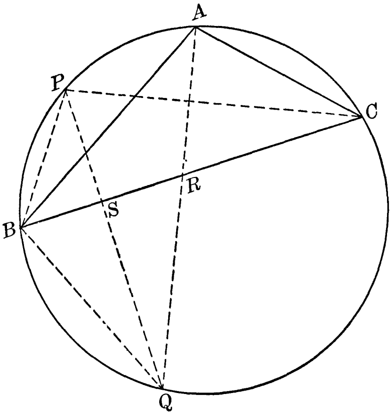
or arc BP = arc PA. Hence

$$\angle\, RSQ = 90^\circ$$

and

$$BS = SR = \tfrac{1}{2}(a - b),$$

$$CS = \tfrac{1}{2}(a + b).$$



Then we have

$$\tan \tfrac{1}{2}(A - B) = \frac{BS}{PS}; \qquad \tan \tfrac{1}{2}(A + B) = \frac{SC}{PS};$$

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{BS}{CS} = \frac{a - b}{a + b}.$$

In addition to the proofs here given I find distinct proofs in R. E. Moritz, *Elements of Plane Trigonometry,* p. 131; Hall and Frink, *Trigonometry,* p. 54; F. Durell, *Plane Trigonometry,* p. 108; in W. F. Cheney, *1920,* 53–54; and in works by the following authors listed *1920,* 53: E. J. Wilczynski, Young and Morgan, and E. W. Hobson.[1]

II. Note on the Law of the Mean.

By H. H. Downing, University of Kentucky.

With certain well-known restrictions on $f(x)$ and its derivative, $f'(x)$, the law of the mean states that

$$\frac{f(x + h) - f(x)}{h} = f'(x + \theta h), \tag{1}$$

where θ is a positive proper fraction.

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[1] Yet another proof is given by G. Lindborg in *Tidskrift för Elementär Matematik, Fysik och Kemi,* vol. 3, May, 1920, p. 201.—Editor.